August 2018. Qualifying exam in analysis

Instructions: Write your name legibly on each sheet of paper. Write only on one side of each sheet of paper. Questions 1-8 are each worth 10 points and question 9 is worth 20 points. The Lebesgue measure will be denoted by λ. The measurability of a real or complex valued function will refer to Lebesgue measurability. If A is a subset of $\mathbb{R}$ and $1 \leq p \leq \infty$, then $\mathcal{L}^p(A)$ and $L^p(A)$ are considered with respect to the Lebesgue measure. You can quote without proof any of the standard theorems covered in Math 703-704, but do indicate why the relevant hypotheses hold. You are also allowed to rely on parts of this exam that you have not solved in order to solve other parts of this exam.

1) Let c_{00} denote the set of infinite sequences of real numbers such that all but at most finitely many terms of each sequence are equal to zero. Equip c_{00} with the metric d defined by

$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \max\{|x_n - y_n| : n \in \mathbb{N}\} \quad \text{for every } (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in c_{00}.$$

You do not have to verify that d is indeed a metric on c_{00}.

a) Is the set $A := \{(x_n)_{n\in\mathbb{N}} \in c_{00} : d((x_n)_{n\in\mathbb{N}}, \vec{0}) \leq 1\}$ closed in (c_{00}, d)? (where $\vec{0}$ denotes the sequence that all its terms are equal to zero).

b) Is the set A (defined in part a)) compact in (c_{00}, d)?

2) Let $f(z) = 1 - y^2 + i(2xy - y^2)$, where $z = x + iy$. Identify the largest open subset D of the complex plane on which f is differentiable. Also compute the derivative, if it exists, of the function f on D.

3) Assume that D is a bounded domain in the complex plane and $f : \overline{D} \to \mathbb{C}$ is a non-constant continuous function which is analytic on D and satisfies $|f(z)| = 1$ for every z on the boundary ∂D of D. Prove that there exists a $z_0 \in D$ such that $f(z_0) = 0$.

4) Let γ be the the curve $|z| = 2$ on the complex plane oriented counterclockwise. Compute $\int_\gamma z^n(1 + i - z)^m dz$ for $n = m = -1$.

5) Let $\mathcal{B}(\mathbb{R})$ denote the σ-algebra of the Borel subsets of $\mathbb{R}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the characteristic function of the set $A = \{(x, y) \in \mathbb{R}^2 : x = y\}$. Let μ denote the counting measure on $\mathcal{B}(\mathbb{R})$. Show that $\int_\mathbb{R}\int_\mathbb{R} f(x, y)d\lambda(x)d\mu(y) \neq \int_\mathbb{R}\int_\mathbb{R} f(x, y)d\mu(y)d\lambda(x)$. Explain why this does not contradict the Fubini Theorem.

6) a) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be called absolutely continuous.

b) Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x \sin \frac{1}{x} & \text{if } x > 0 \end{cases}$$

Check whether f is absolutely continuous.

7) Fix $\alpha \in (0, 1]$. This problem will ask you to compute items associated with the generalized Cantor set C^α. In this notation, the standard Cantor set is C^1 because, on the n^{th} step, one removes sets of length $\frac{1}{3^n}$. The generalized Cantor set C^α is constructed similarly except for, on the n^{th} step, one removes sets of length $\frac{\alpha}{3^n}$. Here are the details of the construction of C^α. Start from $C_0^\alpha = [0, 1]$. For $n \in \mathbb{N}$, assume that a set C_{n-1}^α is given which is the disjoint union of 2^{n-1} many closed intervals, $C_{n-1}^\alpha = \cup_{k=1}^{2^{n-1}} I_{n-1,k}^\alpha$. In the nth step you remove the middle open interval of each set $I_{n-1,k}^\alpha$ of length $\alpha/3^n$. Thus the nth step yields a set C_n^α which is the disjoint union of 2^n many closed intervals: $C_n^\alpha = \cup_{k=1}^{2^n} I_{n,k}^\alpha$. Finally the generalized Cantor set is equal to $C^\alpha = \cap_{n=1}^\infty C_n^\alpha$. The parts below refer to the above construction of the generalized Cantor set and you are required to follow the above notation.

a) For $n \in \mathbb{N}$ compute the Lebesgue measure of $C_{n-1}^\alpha \backslash C_n^\alpha$, i.e. the Lebesgue measure of the set that is removed from C_{n-1}^α in the nth step.

b) Compute the Lebesgue measure of the total set removed in all steps.

c) Compute $\lambda(C^\alpha)$.

d) Let $f : [0, 1] \to [0, 1]$ be the characteristic function of C^α. Find at which points f is discontinuous.

e) For $n \in \mathbb{N} \cup \{0\}$ compute the Lebesgue measure of $C_n^\alpha \backslash C^\alpha$, i.e. the Lebesgue measure of the total set removed from C_n^α in steps $n + 1, n + 2, \ldots$.

f) Find a sequence of Riemann integrable functions on $[0, 1]$ that converges in $L^1[0, 1]$ norm to a function which is not Riemann integrable.

8) a) Prove that if $(X, \mathcal{A}, \mu)$ is a measure space and $f \in \mathcal{L}^1(X, \mathcal{A}, \mu, \mathbb{R})$ then for every $\epsilon > 0$, $\mu(|f| > \epsilon) \leq \frac{1}{\epsilon} \int_X |f| d\mu$.

b) Prove that if $(X, \mathcal{A}, \mu)$ is a measure space and $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{A}$ such that $\sum_n \mu(A_n) < \infty$ then $\mu(\limsup_n A_n) = 0$. Please, also state the definition of $\limsup_n A_n$.

c) Let $(X, \mathcal{A}, \mu)$ be a measure space and $f, f_1, f_2, \ldots$ be real valued measurable functions such that $\sum_n \int |f_n - f| d\mu < \infty$. Prove that $f_n \to f$ μ-a.e. Hint: You can use without a proof the following fact: if $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n < \infty$ then there exist $\epsilon_n > 0$ with $\lim_n \epsilon_n = 0$ such that $\sum_n \frac{a_n}{\epsilon_n} < \infty$.

9) True or False. Prove, or give a counterexample.

a) Every subset of the real line whose Lebesgue outer measure is equal to zero, is Lebesgue measurable.

b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers and for every $n \in \mathbb{N}$ set $s_n := (a_1 + \cdots + a_n)/n$. Then $(a_n)_{n\in\mathbb{N}}$ is Cauchy if and only if $(s_n)_{n\in\mathbb{N}}$ is Cauchy.

c) There exists a linear transformation (also known as Möbius map) which maps the x-axis and y-axis to the lines $y = x$ and $y = 2x$ respectively and leaves the origin fixed.

d) The sequence of functions $(x^n)_{n=1}^{\infty}$ converges uniformly on the open interval $(0, 1)$.

e) If $1 \leq p < \infty$ and $f, g \in \mathcal{L}^p(\mathbb{R})$ then $\|f + g\|_p^p \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p)$ (where $\|\cdot\|_p$ denotes the $\mathcal{L}^p(\mathbb{R})$ norm). Hint: this problem is not related to Holder's inequality.